Notes on Coursera's Game Theory

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Week 01: Introduction and Overview

- Game theory is about self interested agents interacting within a specific set of rules. **Self-Interested Agents** have their own description of states of the world that it likes and acts accordingly. **Utility Functions** are mathematical measures that quantify the degree of preference across alternatives. A theorem by (Von Neumann, 1944) derives the existence of a utility function from more basic assumptions, such as ranking. **Games** are composed by **Players**, **Actions** and **Payoffs**. In what we call the normal form, these are represented by matrices - thus, there is no notion of explicit timing:

$$\begin{bmatrix} (p^1_{11}, p^2_{11}) & (p^1_{12}, p^2_{12}) \\ (p^1_{21}, p^2_{21}) & (p^1_{22}, p^2_{22}) \end{bmatrix}$$

Definition 1 (Best Action) *An agents' i best action a^*_j is yields a better payoff $u_i(a^*_j, a_{-j})$ calculated based on the utility function u_i than any other action A_i he can take, regardless of the actions a_{-j} all the other agents might take.*

$$a^*_i \in BR(a_{-j}) \leftrightarrow \forall a_j \in A_i, u_i(a^*_j, a_{-j}) \geq u_i(a_j, a_{-j})$$

Definition 2 (Nash Equilibrium) *A set of actions - each chosen for a player in the game - is a nash equilibrium if and only if each action is a best action considering all other actions in the set.*

$$a = \langle a_1 \dots a_n \rangle \textit{ is a NE } \leftrightarrow \forall i, a_i \in BR(a_{-i})$$

- We informally define a strategy as choosing an action, or a methodology of choosing some action.

Definition 3 (Strict Domination) *A strategy s_i strictly dominates s'_i if:*

$$\forall s_{-i} \in S_{-i}, u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$$

Definition 4 (Weak Domination) *A strategy s_i weakly dominates s'_i if:*

$$\forall s_{-i} \in S_{-i}, u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}) \quad \exists s_{-i} \in S_{-i}, u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$$

Definition 5 (Very Weak Domination) *A strategy s_i very weakly dominates s'_i if:*

$$\forall s_{-i} \in S_{-i}, u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$$

Definition 6 (Dominant Strategy) *We call a strategy that dominates all other a (strictly or very weak) dominant strategy.*

- Interestingly, if there is a strictly dominant strategy, the nash equilibrium must be unique, otherwise we would prefer both strategies.

Definition 7 (Pareto Dominant) *An outcome o pareto dominates o' if it is atleast as good for every other agent as another outcome o' and strictly better to some agent.*

Definition 8 (Pareto Optimal) *An outcome $o*$ is pareto optimal if no other outcome dominates it.*

Week 02: Mixed Strategy Nash Equilibrium

Definition 9 (Strategy) *A strategy s_i for agent i is a probability distribution over the actions A_i. A **pure strategy** is when only one action is played with positive probability. A **mixed strategy** is when multiple actions have positive probability distributions assigned. Such actions are called the **support** of the strategy. We denote as the set of all strategies for i as S_i, and the set of all strategy profiles to be $S = S_1 \times S_2 \dots S_n$.*

- If players follow a mixed strategy profile $s \in S$, we calculate its **expected** utility: $u_i(s) = \sum_{a \in A} u_i(a) P(a|s)$
- Notice that $P(a|s)$ is the probability that each of the players chose that strategy: $P(a|s) = \prod_{j \in N} s_j(a_j)$
- We can now generalize the idea of best action to the idea of best response, and reframe our definitions:

Definition 10 (Best response) $s_i^* \in BR(s_{-j}) \leftrightarrow \forall s_j \in S_i, u_i(s_j^*, s_{-j}) \geq u_i(s_j, s_{-j})$

Definition 11 (Nash equilibrium for strategies) $s = \langle s_1 \dots s_n \rangle$ *is a NE* $\leftrightarrow \forall i, s_i \in BR(s_{-i})$

Theorem 1 (Nash, 1950) *Every finite game (players and actions) has a Nash equilibrium.*

Example 1 (Calculating Nash Equilibrium on the Battle of Sexes)

$$\begin{array}{cc} & \begin{array}{cc} A_1 & B_1 \end{array} \\ \begin{array}{c} A_2 \\ B_2 \end{array} & \begin{bmatrix} (1,2) & (0,0) \\ (0,0) & (2,1) \end{bmatrix} \end{array}$$

Let player 1 assigns probabilities p to action A_1 and $1-p$ to action B_1. If player 2 better responds with a mixed strategy, then player 1 must make him indifferent between A_2 and B_2 . If he wasn't indifferent, he would increase the probability of one of the actions! We can write that down:

$$\begin{aligned} u_2(A_2, p, 1-p) &= u_2(B_2, p, 1-p) && (1) \\ 2p + 0 * (1-p) &= 1(1-p) + 0p && (2) \\ p &= 1/3 && (3) \end{aligned}$$

We can perform the same calculation for player 2, assuming it assigns probabilities q to action A_2 and $1-p$ to action B_2. Similar calculations yield $q = 2/3$). This implies a Nash Equilibrium: $S = \{(1/3, 2/3), (2/3, 1/3)\}$

- Mixed strategies are used to confuse your opponent and to randomize when uncertain about other's actions. However, they are also a concise decision of what happens in repeated play, and can describe population dynamics, where the game is played multiple times by different people.
- *LCP (Linear Complementarity) Formulation*:

$$\begin{aligned} \sum_{k \in A_2} u_1(a_1^j, a_k^2) * s_2^k + r_1^j = U_1^* \quad \forall j \in A_1 \quad \sum_{j \in A_1} u_2(a_1^j, a_k^2) * s_1^j + r_2^k = U_2^* \quad \forall k \in A_2 \\ \sum_{j \in A_1} s_1^j = 1, \sum_{k \in A_2} s_2^k = 1 \\ s_1^j \geq 0, s_2^k \geq 0, r_1^j \geq 0, r_2^k \geq 0, \quad \forall j \in A_1, \forall k \in A_2 \\ r_1^j * s_1^j = 0, r_2^k * s_2^k = 0 \quad \forall j \in A_1, \forall k \in A_2 \end{aligned} \qquad (4)$$

- *Support Enumerations Formulation*: given a specific support, apply fast solution. Exponential combinations of actions that are potential support, use cleaver heuristics.
- *Complexity*: not NP-complete, there is always a solution! It is NP-complete to find a tiny bit more info than a Nash equilibrium (*e.g. uniqueness*).

Week 03: Alternate Solution Concepts

- *Iterative removal*: Assuming players are rational, we can repeat the following:
  1. Find all strictly dominated strategies;
  2. Remove them, as they won't be played;
- This process **preserves Nash equilibria** and can be used as a preprocessing step before computing an equilibrium. Games solvable by this technique are **dominance solvable**. When there are strictly dominated strategies, order do not matter. If you apply this technique for **weakly dominated strategies**, you can remove best replies, and thus the order does matter. Atleast one equilibria is preserved. Consider the game:

Example 2 (Best reply removal)

$$\begin{array}{cc} & \begin{array}{cc} A_1 & B_1 \end{array} \\ \begin{array}{c} A_2 \\ B_2 \end{array} & \begin{bmatrix} (1,1) & (2,1) \\ (1,2) & (3,1) \end{bmatrix} \end{array}$$

1. *(A_1, A_2) and (B_1, B_2) are pure strategy nash equilibria;*
2. *A_1 weakly dominates $B1$ for player 1 and B_2 weakly dominates A_2 for player 2;*
3. *Depending on how we remove strategies, we loose one of the nash equilibria;*

Definition 12 (Minmax strategy) *The minmax strategy minimizes the other players payoff assuming that they are trying to maximize it. The minmax strategy for player i against $-i$ is:*

$$\arg\min_{s_i} \arg\max_{s_{-i}} u_{-i}(s_i, s_{-i})$$

The minmax value for player i is $\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$. We define the maxmin strategy analogously.

Theorem 2 (von Neumann, 1928) *In any finity, two player, zero-sum game, in any nash equilibrium each player receives a payoff that is equal to both his maxmin value and his minmax value.*

Example 3 (Minmax penalties)

$$\begin{array}{cc} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} L \\ R \end{array} & \begin{bmatrix} (.6,.4) & (.8,.2) \\ (.9,.1) & (.7,.3) \end{bmatrix} \end{array}$$

1. *For the kicker to maximize his minimum:*

   $$max_{s_1} min_{s_2} [s_1(L)s_2(L) * 0.6 + s_1(L)s_2(R) * 0.8 + s_1(R)s_2(L) * 0.9 + s_1(R)s_2(R) * 0.7] =$$
   $$max_{s_1} min_{s_2} [(0.2 - 0.4s_1(L))s_2(L) + 0.1s_1(L) + 0.7]$$
   $$\frac{d}{ds_2}(0.2 - 0.4s_1(L))s_2(L) + 0.1s_1(L) + 0.7 = 0.2 - 0.4s_1(L), 0.2 - 0.4s_1(L) = 0 \implies s_1(L) = s_1(R) = 1/2$$

- For 2 players minmax is solvable with LP:

$$\begin{array}{lll} \text{minimize} & U_1^* & \\ \text{subject to} & \sum_{k \in A_2} u_1(a_1^j, a_2^k) * s_2^k \leq U_1^* \quad \forall j \in A_1, & \sum_{k \in A_2} s_2^k = 1 \\ & s_2^k \geq 0 & \forall k \in A_2 \end{array}$$

- How to model coordination? The current model doesn't model the Battle of the Sexes very well!

Definition 13 (Correlated Equilibrium (Informal)) *A randomized assignment of potentially correlated action recommendations to agents such that nobody wants to deviate.*

Week 04: Extensive Form Games

- Extensive form is an alternative to normal form that makes the temporal structure explicit.

Definition 14 (Perfect-Information Game) *A finite perfect information-game in extensive form is defined by the tuple $(N, A, H, Z, \chi, \rho, \delta, u)$, where:*

1. *Players: N.*
2. *Actions: A.*
3. *Choice nodes: H.*
4. *Action function: $\chi : H \mapsto 2^A$ (what actions can be taken?).*
5. *Player function: $\rho : H \mapsto N$ (who gets the choice?).*
6. *Teminal nodes: Z.*
7. *Successor function: $\delta : H \times A \mapsto U \cup Z$. Maps a choice and an action to a new node s.t. $\forall h_1, h_2 \in H$ and $\forall a_1, a_2 \in A$, if $\delta(h_1, a_1) = \delta(h_2, a_2)$, then $h_1 = h_2$ and $a_1 = a_2$.*
8. *Utility function: $u : Z \mapsto R$.*

Definition 15 (Pure Strategies for PI-EF games) *Let $G = (N, A, H, Z, \chi, \rho, \delta, u)$ be a perfect information extensive-form game. Then the pure strategies of player i consist of the cross product:*

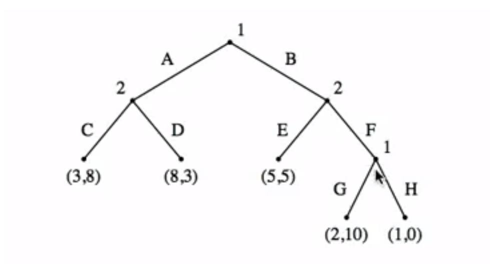
$$\prod_{h \in H, \rho(h) = i} \chi(h)$$

- The definition of **mixed strategies** remain the same: probability distributions over purse strategy. The definition of **best response** remains the same: mixed strategy that maximizes utility given the mixed strategy profile of other agents. The definition of **nash equilibria** remains the same: a strategy profile in which every agent is best responding to every other agent.

- We can convert an extensive form game enumerating the pure strategies in a table: super inneficient computationally. Interestingly, you can't make the conversion backwards! Thus:

Theorem 3 *Every perfect information game in extensive form has a PSNE.*

Example 4 (Motivation subgame perfect equilibrium)



1. *There's something intuitively wrong with the equilibria $(B, H), (C, E)$.*
2. *Why would player 1 ever choose to play H if he got to the second choice node?*
3. *He does so to threaten player 2, to prevent him from choosing F.*

Definition 16 (Subgame of G rooted at h) *The subgame of G rooted at h is the restriction of G to the descendents of H.*

Definition 17 (Subgame of G) *The set subgames of G is defined by the subgames of G rooted at each of the nodes in G.*

Definition 18 (Subgame perfect equilibrium) *s is a subgame perfect equilibrium of G iff for any subgame G' of G, the restriction of s to G' is a Nash equilibrium of G'.*

- Backward induction: we'll go to the leaves and go backward slowly! For zero-sum games this is called the minimax algorithm - you only need to keep one value per node!
- Subgame perfection does not always match the data; Incorrect payoffs? Behavior game theory?

Definition 19 (Imperfect Information games) *A finite imperfect information-game in extensive form is defined by the tuple $(N, A, H, Z, \chi, \rho, \delta, u, I)$, where:*

1. *$(N, A, H, Z, \chi, \rho, \delta, u)$ is a perfect-information extensive-form game, and*
2. *I is an equivalence relation on $\{h \in H : p(h) = i\}$ with the property that $\chi(h) = \chi(h')$ and $\rho(h) = \rho(h')$ whenever there exists a j for which $h \in I_{i,j}$ and $h' \in I_{i,j}$.*

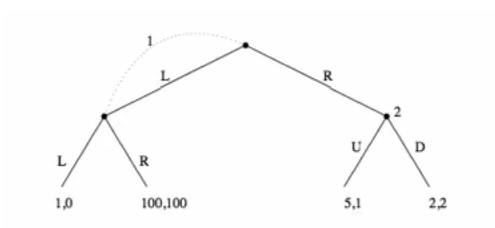
Definition 20 (Pure Strategies for II-EF games) *Let $G = (N, A, H, Z, \chi, \rho, \delta, u, I)$ be a imperfect information extensive-form game. Then the pure strategies of player i consist of the cross product:*

$$\prod_{I_{i,j} \in I_i} \chi(I_{i,j})$$

- Interestingly, we can represent any normal form game, as we can simply ignore the order! We can still do the thing we did before making a imperfect information extensive form game into a normal form game.
- There are two kinds of randomized strategies in imperfect information extensive form games:
  1. mixed strategies: *randomize over pure strategies*;
  2. behavioral strategies: *independent coin toss every time an information set is encountered*;

Theorem 4 *If the game has total recall for every imperfect information game, every behavioral strategy correspond to a mixed strategy.*

Example 5 (Total Recall)



Total recall happens when a player doesn't "forget" any action he took, as in the example above.

- Other concepts of equilibrium such as sequential equilibrium and perfect bayesian equilibrium explicitly models players beliefs on where they are in the tree.

Week 05: Repeated Games

- When players repeatedly play a game, how do we calculate the utility of a strategy?

  Definition 21 (Average reward) *Given an infine seq of payoffs for* i*, his avg reward of is:* $\lim_{k \to \inf} \sum_{j=1}^{k} r_j / k$

  Definition 22 (Future discounted reward) *Given an infine seq of payoffs for* i*, and a discount factor* $\beta, 0 < \beta < 1$*,* i*'s future discounted reward is:* $\sum_{j=1}^{\inf} r_j \beta^j$*. Two possible interpretations for the discount factor is that the agent cares more about his well being in the near term, or that there is a probability* $(1 - \beta)$ *that the game will end in any given round.*

  Definition 23 (Stochastic games) *A stochastic game generalizes the notion a markov decision process (MDP) is a tuple* (Q, N, A, P, R) *where:*

  1. Q *is a finite set of state;* N *is a finite set of players.*
  2. $A = A_1 \times \ldots \times A_n$ *where* A_i *is a finite set of actions available to player* i*.*
  3. $P : Q \times A \times Q \mapsto [0, 1]$ *is the transition probability function, which transitions from state* q *to state* $\hat{q}$ *after a joint action* a*, and*
  4. $R = r_1, \ldots r_n$ *where* $r_i : Q \times A \mapsto R$ *is a real valued payoff function which depends on the state.*

  Definition 24 (Fictitious Plays) *Players mantain explicit belief about the other players:*

  1. *Initialize beliefs about the opponent's strategy; Initialize* $w(a)$ *for every action as your priors.*
  2. *At each turn play best response according to the assessed strategy and observe the oponent's actual play to update your beliefs accordingly; calculate* $\phi(a) = w(a) / \sum_{a' \in A} w(a')$

  Theorem 5 *If the empirical dist* $\forall s_i$ *converges in fictitious play, then it converges to a Nash equilibrium.*

  Theorem 6 *The following are sufficient conditions for the empirical frequencies of play converge in FP: The game is zero sum; The game is solvable by iterated elimination of strictly dominated strategies; The game is a potential game; The game is a* $2 \times n$ *game;*

  Definition 25 (Regret) *The regret an of agent* i *at time* t *for not having played* s *is:* $R^T(s) = \alpha^t - \alpha^t(s)$*.*

  Definition 26 (No-regret learning rule) *A learning rule has no regrets if for any pure strategy of the agent* s*, it holds that* $P([\liminf R^t(s) \leq 0) = 1$*.*

  Definition 27 (Regret Matching) *Choose an action with prob prop. to its regret:* $\phi_i^{t+1}(s) = R^t(s) / \sum_{s' \in S_i} R^t(s')$

  Definition 28 (Pure Stragegy) *A pure strategy in an infinitely-repeated game is a mapping between the entire history* H *and an action* A*.*

- In infinitely-repeated games, Nash's theorem doesn't apply, but we can characterize a set of payoffs that are achievable under equilibrium, without enumerating them!

  Definition 29 *A payoff profile* r *is enforceable if* $r_i \geq v_i$*. Where* $v_i = \min_{s_{-i}} \max_{s_i} u_i(s_{-i}, s_i)$*.*

  Definition 30 *A payoff profile* r *is feasible if there exist rational, non-negative values of* α_a *such that for all* i *we can express* r_i *as* $\sum_{a \in A}$*, such that for all* i*, we can express* r_i *as* $\sum_{a \in A} \alpha_a u_i(a)$*, with* $\sum_{a \in A} \alpha_a = 1$

  Theorem 7 (Folk Theorem for Average Rewards)
  Consider any n-player game G *and any payoff vector* $(r_1, \ldots r_n)$*.*

  1. *If* r *is the payoff in any Nash equilibrium of the infinitely repeated* G *with average rewards, then, for each player* i*,* r_i *is enforceable.*
  2. *If* r *is both feasible and enforceable, then* r *is the payoff in some Nash equilibrium of the infinitely repeated* G *with average rewards.*

- A subgame perfection in the case of a infinitely repeated game means you have nash equilibrium regardless of which point you start on a history. Because of that, repeatedly playing a Nash equilibrium of the stage game is always a subgame perfect equilibrium.

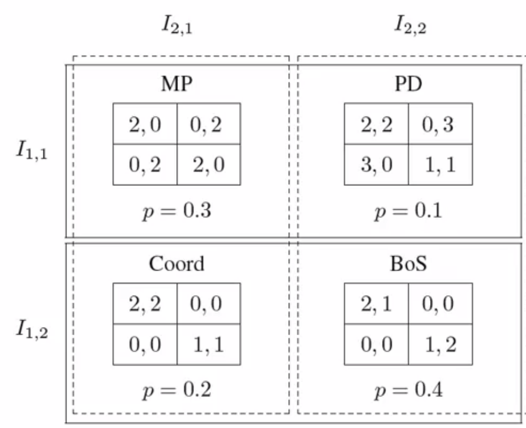
Week 06: Bayesian Games

- So far, we've assumed what everyone knows the number of players, the available actions to each player, and the payoffs associated with each action vector.

  Definition 31 (Bayesian games) *A bayesian game is a tuple (N, G, P, I) where:*

  1. *N a set of agents;*
  2. *G a set of games, where all of them have the same agents and action sets.*
  3. *P is a common prior over games.*
  4. *I is a set of partitions of G, one for each agent.*

  Example 6 (Bayesian game)



  1. *Player 1 knows to tell appart (MP,PD) and (Coord,Bos).*
  2. *Player 2 knows to tell appart (MP,Coord) and (PD,Bos).*

- An alternative yet equivalent definition takes into the consideration the notion of types.

  Definition 32 (Bayesian games) *A bayesian game is a tuple (N, A, Ω, p, u) where:*

  1. *N a set of agents;*
  2. *$A = (A_1, \dots, A_n)$, where A_i is a set one for each player i.*
  3. *$\Theta = (\Theta, \dots, \Theta)$ is a common prior over games.*
  4. *p is a common prior over types.*
  5. *$u = (u_1, \dots, u_n)$ where $u_i : A \times \Theta \mapsto R$ is the utility function for player i*

- In that setting, a pure strat. is a function $s_i : \Theta \mapsto A_i$ which chooses a pure action as a function of his type.
- A mixed strategy is a function $s_i : \Theta \mapsto \prod(A_i)$.
- Some notations of expected utility:
  - *ex-ante* the agent knows nothing about anyone's actual type.
  - *interim* the agent knows her own type, but not others.
  - *ex-post* the agent knows everything.

Week 07: Coalitional Games

- Coalitional game theory approaches how different organizations or individuals cooperate despite the fact they are competing. Our focus is on groups of agents, and thus, given a set of agents, a coalitional game defines how well each coalition can do. We take the payoffs to a coalition as given.
- *Transferable utility assumption:* payoffs may be redistributed among coalition members.

Definition 33 (Coalitional Game with Transferable Utility) *A coalitional game with transferable utility is a pair $G = (N, v)$ where:*

1. *N is a finite set of players indexed by i;*
2. *$v : 2^N \mapsto \mathbb{R}$ associates a real-valued payoff $v(S)$ for each coalition $S \subseteq N$. We assume that $v(\emptyset) = 0$.*

Definition 34 (Superadditive game) *G is superadd.* $\leftrightarrow \forall S, T \subset N, S \cap T = \emptyset \implies v(S \cup T) \geq v(S) + v(T)$

- This implies that the grand coallition has the highest payoff.
- We consider that the grand coallition form and we now focus on its *fairness* and *stability.*
- *Shapley value:* members should receive payments or shares proportional to their marginal contributions. This is tricky, as there may be cases where this is not a simple linear sum.

Definition 35 (Interchangeability) *i and j are interchangeable relative to v if they always contribute the same amount to every coalition of the other agents:*

$$\forall S | i \notin S, j \notin S \implies v(S \cup \{i\}) = v(S \cup \{j\})$$

Axiom 1 (Symmetry) *If i and j are interchangeable, then $\phi_i(N, v) = \phi_j(N, v)$.*

Definition 36 (Dummy player) *i is a dummy player if the amount they i contribute to any coalition is 0:*

$$\forall v \phi_i(N, v) \implies i \textit{ is a dummy player}$$

Axiom 2 (Dummy player) $\forall v, i$ *is a dummy player* $\implies \phi_i(N, v) = 0$

Axiom 3 (Additivity) *For any two v_1 and v_2, $\phi_i(N, v_1 + v_2) = \phi_i(N, v_1) + \phi_i(N, v_2)$ for each i, where the game $(N, v_1 + v_2)$ is defined by $(v_1 + v_2)(S) = v_1(S) + v_2(S)$ for every coalition S.*

Definition 37 *Given a coalitional game N, v the Shapley Value divides payoffs among players according to:*

$$\phi_i(N, v) = \frac{1}{N!} \sum_{S \subset N - \{i\}} |S|!(|N| - |S| - 1)![v(S \cup \{i\}) - v(S)]$$

Theorem 8 (Shapley Value) *Given a coalitional game (N, v), there is a unique payoff division $x(v) = \phi(N, v)$ that divides the full payoff of the grand coalition and that satisfies the Symmetry, Dummy Player and Additivity axioms: the Shapley Value.*

- Would the agents be willing to form the grand coallition given the way it will divide payments? Or would some of them prefer to form smaller coalitions?

Definition 38 (Core) *A payoff vector x is in the core of a coalitional game N, v if and only if:*

$$\forall S \subseteq N, \sum_{i \in S} x_i \geq v(S)$$

- This is analogous to Nash equilibrium, allowing for "group deviations".

Example 7 (Coalitional game)

> **Example (Voting game)**
>
> A parliament is made up of four political parties, A, B, C, and D, which have 45, 25, 15, and 15 representatives, respectively. They are to vote on whether to pass a \$100 million spending bill and how much of this amount should be controlled by each of the parties. A majority vote, that is, a minimum of 51 votes, is required in order to pass any legislation, and if the bill does not pass then every party gets zero to spend.

Is the core always nonempty? ***No***

1. *Consider a voting game where the set of minimal coalitions to meet the required votes is:*

$$\{A, B\}, \{A, C\}, \{A, D\}, \{B, C, D\}$$

2. *If the sum of the payoff to parties B, C, D is less than everything, then this set has incentive to deviate.*
3. *If B, C, D get the entire payoff, A will receive 0 and will have incentive to form a coalition with whichever B, C, D obtained the smallest payoff.*
4. *Thus the core is empty for this game.*

Is the core always unique? ***No***

1. *Consider a voting game where the set of minimal coalitions to meet the required votes is:*

$$\{A, B, C\}, \{A, B, D\}$$

2. *Any payoff vector x that assigns everything to A and B belongs to the core, because there is no way other coalitions would get more value than that.*

Definition 39 (Simple game) *A game $G = (N, v)$ is simple if for all $S \subset N, v(S) \in \{0, 1\}$*

Definition 40 (Veto game) *A player i is a veto player if $v(N - i) = 0$*

Theorem 9 *In a simple game, the core is empty iff there is no vetor player. If there are veto players, the core consists of all payoff vectors in which the nonveto players get 0.*

Definition 41 (Convex Game) *$G = (N, v)$ is convex if for all $S, T \subset N, , v(S \cup T) \geq v(S) + v(T) - v(S \cap T)$.*

Theorem 10 *Every convex game has a nonempty core.*

Theorem 11 *In every convex game, the Shapley value is in the core.*

Example 8 (Example of Convex Game)

> **Example (Airport game)**
>
> Several nearby cities need airport capacity, with different cities needing to accommodate aircraft of different sizes. If a new regional airport is built the cities will have to share its cost, which will depend on the largest aircraft that the runway can accommodate. Otherwise each city will have to build its own airport.
> This situation can be modeled as a coalitional game (N, v), where N is the set of cities, and $v(S)$ is the sum of the costs of building runways for each city in S minus the cost of the largest runway required by any city in S.